EFFECTIVE APPROXIMATE SOLUTIONS OF TWO MIXED PROBLEMS OF STEADY-STATE HEAT CONDUCTION UNDER CONDITIONS OF CONVECTIVE HEAT TRANSFER

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Two methods of solving mixed problems of steady-state heat conduction under conditions of convective heat transfer are considered: successive approximations and superposition. The results of a calculation of the dimensionless temperature by both methods are presented.

Using plane and axially-symmetrical problems as examples, we shall consider two approximate methods of solving mixed problems of steady-state heat conduction subject to convective heat transfer. One of these is based on seeking the solution in the form of a series in powers of the parameters entering into the boundary conditions, while the second is based on the superposition of solutions relating to auxiliary problems. The first of these methods will subsequently be called the method of successive approximations; it enables us to construct as accurate a solution as desired for values of the parameter greater than unity; the second (superposition) method is suitable for any arbitrary value of the parameter, and gives an accuracy sufficient for engineers' calculations.

1. Let us consider the half plane $y > 0$; on part of the boundary of this half plane the conditions of convective heat transfer are specified for $|x| < 1$, while on the other half ($|x| > 1$) the temperature equals zero. Calculation of the steady-state thermal field amounts in this case to seeking the harmonic function $\vartheta(x, y)$ satisfying the boundary conditions

$$\vartheta - k\frac{\partial\vartheta}{\partial y} = 1 \text{ for } y = 0,\ |x| < 1, \tag{1}$$

$$\vartheta = 0 \text{ for } y = 0,\ |x| > 1, \tag{2}$$

where $k = 1/\mathrm{Bi} = \mathrm{const} > 0$.*

Assuming that $k > 1$, we shall seek an expression for the temperature in the form

$$\vartheta = \sum_{m=1}^{\infty} \vartheta_m k^{-m}, \tag{3}$$

where each of the functions ϑ_m is assumed to be harmonic. Substituting Eq. (3) into the boundary conditions (1), (2), we find that the boundary conditions for all the functions ϑ_m at $y = 0$ may be written

$$\left.\frac{\partial\vartheta_m}{\partial y}\right|_{|x|<1} = \begin{cases} -1 & \text{for } m = 1, \\ \vartheta_{m-1} & \text{for } m > 1, \end{cases} \tag{4}$$

$$\vartheta_m|_{|x|>1} = 0.$$

*In [1, 2] this problem is reduced to the solution of a Fredholm integral equation of the second kind; an approximate solution by the method of characteristic surfaces giving satisfactory accuracy for small values of the parameter k is also considered in [1].

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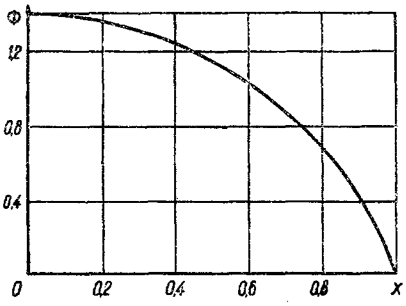

Fig. 1. Graph of the function $\Phi(x)$.

For these boundary conditions the solution to the problem of determining ϑ_1 is quite well known and yields the following [3, 4]:

$$\vartheta_1 \Big|_{\substack{|x|<1 \\ y=0}} = \sqrt{1-x^2}. \tag{5}$$

Seeking the second approximation in the form

$$\vartheta_2 = \int_0^\infty A(\lambda) \exp(-\lambda y) \cos \lambda x d\lambda, \tag{6}$$

we reduce it to the solution of the following system of paired integral equations:

$$\int_0^\infty A(\lambda) \lambda \cos \lambda x d\lambda = -\sqrt{1-x^2} \quad \text{for} \quad |x| < 1, \tag{7}$$

$$\int_0^\infty A(\lambda) \cos \lambda x d\lambda = 0 \quad \text{for} \quad |x| > 1. \tag{8}$$

Integrating Eq. (7) and, differentiating (8) with respect to x, we bring the system to the form

$$\int_0^\infty A(\lambda) \sin \lambda x d\lambda = F(x) \quad \text{for} \quad |x| < 1, \tag{9}$$

$$\int_0^\infty \lambda A(\lambda) \sin \lambda x d\lambda = 0 \quad \text{for} \quad |x| > 1,$$

where

$$F(x) = -\frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin x \right). \tag{10}$$

The solution of the system (9) was obtained in general form in [5]. For the F(x) defined by Eq. (10) we have

$$A(\lambda) = -\frac{2}{\pi} \int_0^1 t\,\mathrm{E}(t) J_0(\lambda t)\, dt,$$

$$\vartheta_2 \Big|_{\substack{|x|<1 \\ y=0}} = -\frac{2\Phi(x)}{\pi}, \tag{11}$$

where

$$\Phi(x) = \int_0^{\sqrt{1-x^2}} \mathrm{E}\left(\sqrt{z^2+x^2}\right) dz \quad \text{(see Fig. 1)}.$$

In the particular case in which x = 0

$$\vartheta_2 \Big|_{x=y=0} = -\frac{2}{\pi}\left(\frac{1}{2} + G\right), \tag{12}$$

where G = 0.916....

We may calculate the next approximation in an analogous way; the number of these will determine the accuracy of the resultant solution (the sign of the errors associated with successive approximations will be alternating, as may readily be verified).

Using Eq. (3) and the expressions for ϑ_1 and ϑ_2, we may find an approximate expression for the temperature at any point of the boundary surface. In particular

$$\vartheta \Big|_{x=y=0} \approx \frac{1}{k} - \frac{2.8}{\pi k^2}. \tag{13}$$

2. Let us consider the solution of the problem subject to conditions (1), (2) by the superposition method [7]. Following this method we express the unknown function $\vartheta(x, y)$ as a sum of two functions

$$\vartheta = \vartheta' + \vartheta'',$$

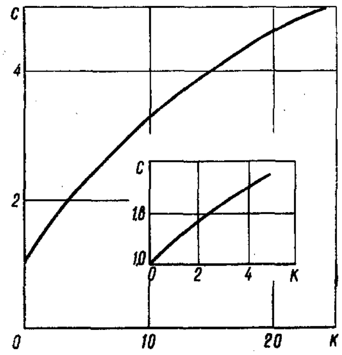

Fig. 2. Graph to determine the parameter c.

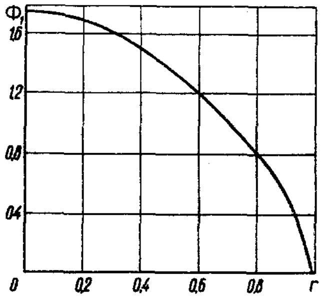

Fig. 3. Graph of the function $\Phi_1(r)$.

satisfying the following boundary conditions at y = 0:

$$\vartheta' = \begin{cases} D, & |x| < c, \quad (14) \\ 0, & |x| > c, \quad (15) \end{cases}$$

$$\vartheta'' = f(x) = \begin{cases} -D, & 1 \leqslant |x| < c, \quad (16) \\ 0, & |x| > c, \quad (17) \end{cases}$$

$$\frac{\partial \vartheta''}{\partial y} = 0, \quad |x| < 1, \tag{18}$$

where D and c are certain parameters.

The boundary condition (2) is satisfied identically for any values of the parameters D and c. We require that condition (1) should be satisfied at the points* x = 0 and x = 1; then the parameters D and c will be determined by simultaneous solution of the equations

$$\vartheta''(0,\ 0) - k\frac{\partial \vartheta'(0,\ 0)}{\partial y} = 1 - D, \tag{19}$$

$$1 + k\frac{\partial \vartheta'(1,\ 0)}{\partial y} = 0. \tag{20}$$

Finding ϑ' and ϑ'' causes no serious difficulty, the expression for the first of them already being known [4]:

$$\vartheta' = \frac{D}{\pi}\left(\operatorname{arctg}\frac{c+x}{y} + \operatorname{arctg}\frac{c-x}{y}\right). \tag{21}$$

Seeking the expression for ϑ'' in the ordinary form

$$\vartheta''(x,\ y) = \int_0^\infty B(\lambda)\exp(-\lambda y)\cos\lambda x d\lambda, \tag{22}$$

we arrive at a system of integral equations

$$\int_0^\infty B(\lambda)\cos\lambda x d\lambda = f(x), \quad x > 1, \tag{23}$$

$$\int_0^\infty \lambda B(\lambda)\cos\lambda x d\lambda = 0, \quad x < 1. \tag{24}$$

Integrating (24) with respect to x, we reduce the system in question to the form

$$\int_0^\infty B(\lambda)\cos\lambda x d\lambda = f(x), \quad x > 1, \tag{25}$$

$$\int_0^\infty B(\lambda)\sin\lambda x d\lambda = 0, \quad x < 1. \tag{26}$$

*Since the solution of the problem in question is an even function in x, we shall subsequently (in order to be specific) assume that x > 0.

The function $B(\lambda)$ we shall seek in the form

$$B(\lambda) = \int_1^\infty \psi(t) J_0(\lambda t)\, dt. \tag{27}$$

Here Eq. (26) is satisfied identically, while (25) is reduced to the Schlemilch integral equation

$$\int_0^\xi \frac{\bar{\psi}(\tau)\, d\tau}{\sqrt{\xi^2 - \tau^2}} = \bar{f}(\xi), \tag{28}$$

where $\xi = 1/x$; $\tau = 1/t$; $\bar{\psi}(\tau) = \psi(1/\tau)/\tau$; $\bar{f}(\xi) = f(1/\xi)/\xi$.

The solution to Eq. (28) is already well known [8]. Starting from (16), (17), (22), (27), and (28) we have

$$\vartheta'' = -\frac{2D}{\pi} \int_1^\infty \frac{tdt}{\sqrt{c^2 - t^2}} \int_0^\infty \exp(-\lambda y) J_0(\lambda t) \cos \lambda x d\lambda. \tag{29}$$

Using (19) and (20), we obtain relations connecting the parameters D and c to the specified quantity k:

$$k = \frac{c(c^2 - 1)}{2} \arccos \frac{c^2 - 2}{c^2}, \tag{30}$$

$$D = \frac{\pi (c^2 - 1)}{2kc}. \tag{31}$$

It may be shown that the results are valid for any value of $k \in [0, \infty)$.

Using (21) and (29)-(31) we find the following equations for the temperature and its normal derivative, depending on the parameter c (see Fig. 2):

$$\vartheta(0, y) = \frac{\pi}{c^2 \arccos \frac{c^2 - 2}{c^2}} \left(\frac{2}{\pi} \operatorname{arctg} \frac{c}{y} + \frac{1}{\pi} \arccos \frac{c^2 - y^2 - 2}{c^2 + y^2} - 1 \right), \tag{32}$$

$$\vartheta(x, 0)|_{x<1} = \frac{\arccos \frac{c^2 + x^2 - 2}{c^2 - x^2}}{c^2 \arccos \frac{c^2 - 2}{c^2}}, \tag{33}$$

$$\left.\frac{\partial \vartheta(x, 0)}{\partial y}\right|_{x<1} = -\frac{2}{\sqrt{c(c^2 - x^2)} \arccos \frac{c^2 - 2}{c^2}}. \tag{34}$$

Starting from (34) we also obtain an expression for the total thermal flux of the system

$$Q = -2 \int_0^1 \frac{\partial \vartheta(x, 0)}{\partial y} dx = \frac{4 \operatorname{Arc\,th} c}{c^2 \arccos \frac{c^2 - 2}{c^2}},$$

which may also be used to calculate the thermal resistance per unit length.

The error in the resultant solution is determined by the deviation of the approximate boundary condition for $x < 1$ from the condition specified by (1); calculations show that this error is no more than a few percent of the maximum value of the function at the boundary.

3. These two methods may be applied not only to plane but also to axially symmetrical problems with boundary conditions of the third kind.

Let us consider the space $z > 0$; on part of the boundary of this space (inside a circle of unit radius) the conditions of convective heat transfer are satisfied, while on the remaining part of the boundary plane the temperature equals zero. A calculation of the temperature distribution in this case reduces to finding $\vartheta(r, z)$ subject to the following boundary conditions*

$$\vartheta - k \frac{\partial \vartheta}{\partial z} = 1 \text{ for } z = 0,\ r < 1, \tag{35}$$

$$\vartheta = 0 \text{ for } z = 0,\ r > 1. \tag{36}$$

*In [9] this problem is reduced to a solution of a Fredholm equation of the second kind.

The solution to this problem by the superposition method is given in [7]. In the present case we shall therefore confine ourselves to the method of successive approximations. Proceeding as in the case of the plane problem, i.e., seeking the solution for k > 1 in the form of a series (3), for z = 0 we arrive at boundary conditions for the functions ϑ_m analogous to conditions (4):

$$\left.\frac{\partial\vartheta_m}{\partial z}\right|_{\substack{z=0\\ r<1}} = \begin{cases} -1 & \text{for } m=1, \\ \vartheta_{m-1} & \text{for } m>1, \end{cases} \tag{37}$$

$$\vartheta_m\big|_{\substack{z=0\\ r>1}} = 0.$$

The solution to the problem of determining ϑ_1 is already known [4]:

$$\vartheta_1\Big|_{\substack{z=0\\ r<1}} = \frac{2}{\pi}\sqrt{1-r^2}. \tag{38}$$

Seeking the second approximation in the form

$$\vartheta_2 = \int_0^\infty A(\lambda)\exp(-\lambda z)\,J_0(\lambda r)\,d\lambda, \tag{39}$$

and allowing for (37) and (38), we arrive at the following system of integral equations

$$\int_0^\infty \lambda A(\lambda)\,J_0(\lambda r)\,d\lambda = -\frac{2}{\pi}\sqrt{1-r^2} \ \text{for}\ r<1, \tag{40}$$

$$\int_0^\infty A(\lambda)\,J_0(\lambda r)\,d\lambda = 0 \ \text{for}\ r>1.$$

The solution to a system of the form (40) was obtained in [7]; in the present case it leads to the following expression for ϑ_2 at the boundary:

$$\vartheta_2\Big|_{\substack{z=0\\ r<1}} = -\frac{2\Phi_1(r)}{\pi^2}, \tag{41}$$

where

$$\Phi_1(r) = \int_r^1\left[t + \frac{1}{2}(1-t^2)\ln\frac{1+t}{1-t}\right]\frac{dt}{\sqrt{t^2-r^2}} \quad \text{(see Fig. 3).}$$

At r = 0 the function $\Phi_1(r)$ may be expressed in explicit form; in the second approximation the formula for the temperature becomes

$$\vartheta\Big|_{r=z=0} \approx \frac{2}{\pi k} - \frac{4+\pi^2}{4\pi^2 k^2}. \tag{42}$$

TABLE 1. Results of a Calculation of the Dimensionless Temperature

k			2	5	10
$\vartheta\|_{x=y=0}$ (plane problem)	by successive approximation	first approximation	0,50	0,20	0,10
		second approximation	0,27	0,16	0,09
	by superposition		0,35	0,17	0,09
$\vartheta\|_{r=z=0}$ (axially symmetrical problem)	by successive approximation	first approximation	0,32	0,13	0,06
		second approximation	0,22	0,11	0,06
	by superposition		0,26	0,11	0,06

4. Comparing the results of calculations carried out by the two foregoing methods for the dimensionless temperature we find excellent agreement between them for both the plane and the axially symmetric problem (Table 1).

For large values of k both methods lead to the same approximate equations.

For the plane problem

$$\vartheta|_{x=y=0} \approx \frac{1}{k} .$$

For the axially symmetric problem

$$\vartheta|_{r=z=0} \approx \frac{2}{\pi k} .$$

The results here presented may also be used for solving other analogous problems in potential theory, in particular in calculating the electric field of linearly-polarized electrodes.

NOTATION

ϑ, Q	are the dimensionless temperature and thermal flux;
Bi,	is the Biot number;
k	is the boundary-condition parameter;
f, F, φ, Φ, ψ, A, B	are the symbols of the functions;
x, y	are the Cartesian coordinates;
r, z	are the cylindrical coordinates;
t, τ	are the integration variables;
λ	is the variable separation parameter in the Laplace equation;
c, D	are the parameters;
E	is the complete elliptic integral of the second kind;
J_0	is the Bessel function of the first kind and zeroth order;
G	is the Catalan constant.

LITERATURE CITED

1. Yu. Ya. Iossel' and R. A. Pavlovskii, Inzh.-Fiz. Zh., 10, No. 2 (1966).
2. Yu. N. Kuz'min, Zh. Tekh. Fiz., 36, No. 8 (1966).
3. R. R. Dogonadze, V. G. Levich, and Yu. A. Chizmadzhev, Zh. Fiz. Khim., 33, No. 5 (1959).
4. Yu. Ya. Iossel', É. S. Kochanov, and N. G. Strunskii, Questions of Calculating and Modeling the Electrochemical Anticorrosion Protection of Ships, Sudostroenie, Leningrad (1965).
5. R. A. Pavlovskii, Inzh.-Fiz. Zh., 7, No. 11 (1964).
6. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products [in Russian], Fizmatgiz (1963).
7. R. A. Pavlovskii, Inzh.-Fiz. Zh., 15, No. 2 (1968).
8. E. G. Whittaker and D. N. Watson, Course of Modern Analysis, Part 1 [Russian translation], Fizmatgiz (1963).
9. Yu. N. Kuz'min, Zh. Tekh. Fiz., 36, No. 2 (1966).